Stabilizing model predictive control for constrained nonlinear distributed delay systems

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A R T I C L E   I N F O

Article history:
Received 18 October 2009
Received in revised form 21 November 2010
Accepted 16 December 2010
Available online 12 January 2011

Keywords:
Stabilizing model predictive control
Distributed delay
Nonlinear systems

A B S T R A C T

In this paper, a model predictive control scheme with guaranteed closed-loop asymptotic stability is proposed for a class of constrained nonlinear time-delay systems with discrete and distributed delays. A suitable terminal cost functional and also an appropriate terminal region are utilized to achieve asymptotic stability. To determine the terminal cost, a locally asymptotically stabilizing controller is designed and an appropriate Lyapunov–Krasovskii functional of the locally stabilized system is employed as the terminal cost. Furthermore, an invariant set for locally stabilized system which is established by using the Razumikhin Theorem is used as the terminal region. Simple conditions are derived to obtain terminal cost and terminal region in terms of Bilinear Matrix Inequalities. The method is illustrated by a numerical example.

1. Introduction

The dynamic models of many phenomena in engineering and biology involve severe nonlinearities and significant time-delays. The presence of time-delays brings challenges in the analysis and control of these systems [1,2]. Moreover, delay differential equations with continuous delay deviation are specified as distributed delay systems and appear for instance in the modeling of traffic flow [3] and the combustion chamber of a liquid rocket motor [4].

Model predictive control (MPC), also known as the receding horizon control, is a popular control strategy because of its many advantages including constraints handling. In this procedure, based on the current measurement, a finite horizon optimal control problem is solved. Then, the computed control is applied to the system until the next measurement becomes available. However, in the predictive control scheme the closed-loop stability is not guaranteed naturally and is highly dependent on the design of the finite horizon optimal control problem. Therefore, many approaches have been proposed to avoid this problem in delay-free systems. Most of these methods utilize terminal state constraints, often together with a terminal cost, to ensure closed-loop system stability using a finite horizon length [5–8].

A stabilizing model predictive control for linear state-delayed systems and linear input-delayed systems was developed in [9,10], respectively. In [11], to assure the closed-loop stability in the predictive control of nonlinear time-delay systems, an expanded zero terminal state constraint which takes the past trajectory of the system into account was utilized in the finite horizon problem. This additional constraint leads to a feasibility problem because the system has to be steered to the zero equilibrium state in finite time. Furthermore, this equality constraint increases the amount of online computations required for optimization. In [12], for a special class of unconstrained nonlinear time-delay systems, a predictive controller with guaranteed stability was presented. In this method, the closed-loop stability is achieved via a suitably defined terminal cost functional. In [13], a predictive control scheme with guaranteed closed-loop stability was suggested for more general unconstrained nonlinear distributed delay systems utilizing the Control Lyapunov–Krasovskii Functional (CLKF). In [14], using an appropriate terminal cost functional and terminal inequality constraint in the finite horizon problem, the closed-loop stability was guaranteed for a special class of nonlinear time-delay systems. It was assumed that the state rate is restricted by a linear bound. This strong assumption on system dynamics makes this approach ineffective. Moreover, a globally stabilizing control law is needed to determine the terminal cost functional. In [15], motivated by the general framework to design stabilizing predictive controllers, we proposed a stabilizing predictive control approach for an input constrained nonlinear time-delay system with discrete delay. In this approach, an appropriate terminal cost functional and a suitably defined terminal region are included in the finite horizon optimal control problem, in order to guarantee closed-loop stability.

To the authors’ best knowledge the most general result in the predictive control of constrained nonlinear time-delay systems [15] was developed for discrete delay systems. In this paper, we design a model predictive controller with guaranteed closed-loop
stability for input constrained nonlinear time-delay systems with discrete and distributed delays. Utilizing Lyapunov–Krasovskii and Lyapunov–Razumikhin Theorems [16], simple conditions are derived in terms of bilinear matrix inequalities (BMIs) [17] to determine off-line the terminal cost and region.

The rest of this paper is organized as follows. The problem setup is described in Section 2. In Section 3, the main results which supply a stabilizing predictive control scheme for nonlinear time-delay systems with input constraints is explained. In Section 4, a procedure to determine the terminal cost and the terminal region is proposed. Section 5 illustrates the performance of the proposed controller by an example. Finally, in Section 6, some concluding remarks are presented.

Notation: Euclidean norm of n-dimensional vectors in \( \mathbb{R}^n \) is denoted by \( \| \cdot \| \). Induced 2-norm of matrix \( P \) is symbolized by \( \| P \| \). \( \lambda_{\max}(P) \) and \( \lambda_{\min}(P) \) refer to the maximum and minimum eigenvalues of matrix \( P \), respectively. For \( \tau > 0 \), \( C_\tau = \{ (t, -\tau, 0) \} \), \( R^\tau \) denotes the space of continuous functions mapping the interval \([-\tau, 0]\) into \( R^n \). The norm on \( C_\tau \) is defined as \( \| \psi \|_{C_\tau} = \sup_{-\tau \leq s \leq 0} \| \psi(s) \| \).

2. Problem setting

Consider an affine nonlinear time-delay system with discrete and distributed delays described by:

\[
\dot{x}(t) = f(x(t), x(t - \tau_1), \ldots, x(t - \tau_n)) + \int_{-\tau}^{0} G(\theta)F(x(t), x(t - \tau_1), \ldots, x(t - \tau_m), x(t + \theta))d\theta + g(x(t)), x(t - \tau_1), \ldots, x(t - \tau_n) u(t) \tag{1}
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector and \( u(t) \in U \subset \mathbb{R}^r \) is the input vector. The constraint set \( U \) is compact and contains the origin. Time delays \( \tau_i \) and the time instant \( t \) are known and \( \varphi(t) \) is \( C_\tau \) is the initial vector function, for \( t = [-\tau, 0] \). \( f \), \( G \), \( F \) and \( g \) are bounded and continuously differentiable vector functions of their arguments and without loss of generality, we suppose that \( F(x(t), x(t - \tau_1), \ldots, x(t - \tau_m)) = 0 \). Moreover, the origin \( 0 \in C_\tau \) is assumed to be the equilibrium state of the system. The problem of interest is to asymptotically stabilize the origin of system (1) via model predictive control.

In the model predictive control strategy, based on measurements obtained at time \( t \), the controller predicts the future behavior of the system over a prediction horizon \( T \) and determines the control input such that a predetermined cost functional \( J \) is minimized. The obtained control function will be implemented only until the next sampling time. It is well-known that an incisive definition of the finite horizon optimal control problem may cause terrible consequences especially if the horizon is too short. Even though the resulting optimal cost may be well-behaved, the closed-loop system may be unstable. To guarantee closed-loop stability, certain conditions in the finite horizon optimization problem have to be met to assure that the associated optimal cost can be used as a Lyapunov-like function for a closed-loop system.

3. Stabilizing predictive controller

In the proposed scheme, first, a locally asymptotically stabilizing controller is designed in some neighborhood \( \Omega \) of the origin, and then using this controller an upper bound of the infinite horizon cost-to-go is computed and used as a terminal cost. Furthermore, a constraint is added to the finite horizon optimal control problem that requires the final state to lie within the \( \Omega \).

The finite horizon optimal control problem at time \( t \) is formulated as follows:

\[
\min_{u} J(x(t), u, t, T) = \int_{t}^{t+T} q(x(t'), u(t'))dt' + V(x(t+T)) \tag{2}
\]

subject to

\[
\dot{x}(t') = f(x(t'), x(t' - \tau_1), \ldots, x(t' - \tau_n)) + \int_{-\tau}^{0} G(\theta)F(x(t'), x(t' - \tau_1), \ldots, x(t' - \tau_m), x(t' + \theta))d\theta + g(x(t')), x(t' - \tau_1), \ldots, x(t' - \tau_n) u(t') \]

\[
x_c = \bar{x}(t') \in \Omega, \quad \bar{u}(t') \in U \]

where \( (x(t')) \) is the predicted trajectory starting from the real state \( x_c = \bar{x}(t + \theta) = \max(x(t, \theta), \theta) \leq \theta \leq 0 \) and driven by \( \bar{u}(t') \) for \( t' \in [t, t + T] \). Terminal region \( \Omega \) is a closed set and contains \( 0 \in C_\tau \), \( V \) is a suitably defined terminal cost functional. Continuous stage cost \( q : R^r \times R^r \rightarrow R^+ \) penalizes states and control inputs according to:

\[
q(x(t), u(t)) \geq c_q(x(t)^2 + |u(t)|^2), \quad \forall t \in [t, t + T] \tag{4}
\]

for \( c_q > 0 \) and \( q(0,0) = 0 \). It is assumed that the optimal control which optimizes \( J(x_{c}, u_{c}, T) \) is given by \( \bar{u}(t') \) and also \( J(x_{c}, u_{c}, t, T)_{u = u_{c}} = J^*(x_{c}, t, T) \) is the associated optimal cost. The control input to the system \( \bar{u}(t') \) is defined by the optimal solution of problem at the sampling instants \( t = k\Delta, k = 0, 1, 2, \ldots \) as follows:

\[
\bar{u}(t') = \hat{u}(t'), \quad t' \leq t' + T \tag{5}
\]

The implicit feedback controller resulting from the application of the (5) is asymptotically stabilizing provided that the design parameters: time horizon \( T \), terminal cost \( V \) and terminal region \( \Omega \) satisfy the following crucial conditions:

(1) for nonlinear time-delay system (1) a locally asymptotically stabilizing controller \( u^*(t) = k(x^*) \in U \) is designed and a continuously differentiable positive definite function \( V(x^*) \) is determined such that \( \forall x_c \in \Omega \):

\[
\frac{d}{dt} V(x^*) \leq -q(x(t)), k(x(t)) \tag{6}
\]

(2) the time horizon \( T \) is chosen such that the set \( \Omega \) is reachable from initial state with admissible control, i.e. Finite horizon problem (2)-(3) admits a feasible solution at \( t = 0 \).

The next section is dedicated to describe the derivation of a systemized procedure to ensure the above conditions for the system (1). But first, it is shown that the proposed predictive controller stabilizes system (1) provided appropriate selection of design parameters. Before the asymptotic stability of the closed-loop system is addressed, we need to show that a repeated solution of the finite horizon optimal control problem (2)-(3) is feasible. Feasibility of the problem implies that there exists one control input \( u(t') \), \( t' \in [t, t + T] \) that results in a bounded value of the problem cost and satisfies all the constraints of the problem.

Proposition 1. Finite horizon problem (2)-(3) is feasible for every \( t > 0 \) if it admits a feasible solution at \( t = 0 \).

Proof. Suppose that at time \( t \), a feasible solution of (2)-(3) exists, i.e. \( u(t'), t' \in [t, t + T] \). At the next time instant \( t + \delta, \delta > 0 \) a feasible control input can be constructed as follows:

\[
\hat{u}(t') = \begin{cases} \bar{u}(t') & t' \in [t + \delta, t + T] \\
 k(x_{c}) & t' \in [t + T, t + T + \delta]. \end{cases}
\]

As can be seen \( \hat{u}(t') \) is concatenated with a part of feasible control that steers \( x_{c, t+\delta} \) to \( x_{c, t+T} \in \Omega \) where \( k(x_{c}) \) keeps the system trajectory for \( t + T \leq t' \leq t + \delta \) at \( \Omega \) while satisfying all constraints. So, the feasibility of (2)-(3) at time \( t \) results in feasibility at time \( t + \delta \). Since the finite horizon problem (2)-(3) admits a feasible solution at \( t = 0 \), by induction it is feasible for every \( t > 0 \). □
To establish the asymptotic stability proof, first, in the Proposition 2, it will be shown that the optimal cost of problem (2)–(3) is non-increasing. Then in Proposition 3, it will be proved that the non-increasing property of the optimal cost is sufficient for the asymptotic stability of the proposed predictive controller. The combination of Propositions 2 and 3 enables us to state our final result about the asymptotic stability of the proposed scheme.

Proposition 2. Suppose that the stability conditions (1) and (2) hold, then the optimal cost \( J^*(x, t, T) \) of finite horizon optimal control problem (2)–(3), satisfies the following non-increasing property: \( J^*(x, t, T + \delta) \leq J^*(x, t, T) \).

Proof. The cost function for time interval \([t, t + T + \delta]\), beginning at initial state \( x \) is \( J(x, \mathbf{u}, t, T + \delta) \). According to (2)–(3), one has:

\[
J^*(x, t, T + \delta) \leq J(x, \mathbf{u}, t, T + \delta) = J(x, \mathbf{u}, t, T + \delta) + \int_{t}^{t+T+\delta} q(x(t'), \mathbf{u}(t')) dt' + V(x_{t+T+\delta})
\]

On the other hand, condition (1) results:

\[
-V(x_{t+T}) + V(x_{t+T+\delta}) + \int_{t+T+\delta}^{t+T+\delta} q(x(t')) dt' \leq 0
\]

hence, we can conclude that: \( J^*(x, t, T + \delta) \leq J^*(x, t, T) \) which completes the proof.

Proposition 3. If the optimal cost of problem (2)–(3) satisfies \( J^*(x, t, T + \delta) \leq J^*(x, t, T) \) for every \( \delta > 0 \), then the system (1) with the predictive control law (5) is asymptotically stable.

Proof. Since \( J^*(x, t, T + \delta) \leq J^*(x, t, T) \), the following is obtained:

\[
J^*(x, t, T) = \int_{t}^{T} q(x(t'), \mathbf{u}(t')) dt' + V(x_{t+T})
\]

\[
= \int_{t+\delta}^{T} q(x(t'), \mathbf{u}(t')) dt' + \int_{t}^{t+\delta} q(x(t'), \mathbf{u}(t')) dt' + J^*(x_{t+\delta}, t + \delta, T - \delta)
\]

\[
\geq \int_{t}^{t+\delta} q(x(t'), \mathbf{u}(t')) dt' + J^*(x_{t+\delta}, t + \delta, T).
\]

Rearranging the above inequality and dividing both sides by \( \delta \) yields:

\[
J^*(x_{t+\delta}, t + \delta, T) - J^*(x, t, T) \leq -\frac{1}{\delta} \int_{t}^{t+\delta} q(x(t'), \mathbf{u}(t')) dt'.
\]

Now, if \( \delta \to 0 \), using the inequality (4), yields to:

\[
\frac{d}{dt} J^*(x, t, T) \leq -q(x, \mathbf{u}) \leq -c_q(|x|^2 + |\mathbf{u}|^2),
\]

which means that \( J^*(x, t, T) \) is non-increasing. Since \( J^* \geq 0 \) then \( J^* \to c \) as \( t \to \infty \), where \( c \) is a nonnegative constant, and so \( dJ*/dt \to 0 \) as \( t \to \infty \). Hence, it is clear that \( x(t), \mathbf{u}(t) \to 0 \) as \( t \to \infty \). The final step to complete the stability analysis is to show the optimal cost \( J^*(x, t, T) \) is continuous at \( x = 0 \). Let \( \phi \) belongs to some neighborhood of the origin and \( \phi \neq 0 \).

Now, let us focus on the following system for \( t' \in [t, t + T] \):

\[
\dot{x}(t') = f(x(t'), x(t' - \tau_1), \ldots, x(t' - \tau_m)) + \int_{t-\theta}^{t} G(\theta)F(x(t'), x(t' - \tau_1), \ldots, x(t' - \tau_m)) \dot{x}(t' + \theta) d\theta
\]

\[
x(t + \theta) \dot{x}(t) + g(x(t), x(t - \tau_1)) \mathbf{u}(t)
\]

Since \( f \) and \( F \) are smooth functions of their arguments, a solution \( x(t) \) exists on \([t, t + T]\) and is continuous in the initial state at \( x_0 = 0 \) [16]. The mentioned neighborhood can be selected such that the terminal region condition is satisfied. Now, let the associated cost functional be denoted by following:

\[
J(\phi, t, t + T) = \int_{t}^{t+T} q(x(t'), 0) dt' + V(x_{t+T}).
\]

Now, based on Propositions 2 and 3 the main result of this section can be stated as Theorem 1.

Theorem 1. Assume that in the finite horizon optimal control problem (2)–(3), design parameters are selected such that conditions (1) and (2) hold. Then, the closed-loop system resulting from the application of the predictive control strategy to system (1) is asymptotically stable.

As mentioned, terminal cost \( V \) and the terminal region \( \Omega \) are determined off-line such that the closed-loop performance over the infinite horizon is approximated. To demonstrate this fact, we consider an infinite horizon cost defined by (7) which can be separated into two parts:

\[
J^*_\infty(x, t) = \min_{\mathbf{u}} \int_{t}^{\infty} q(x(t'), \mathbf{u}(t')) dt'
\]

\[
\leq \min_{\mathbf{u}} \left\{ \int_{t}^{t+T} q(x(t'), \mathbf{u}(t')) dt' + \int_{t+T}^{\infty} q(x(t'), k(x(t')) dt' \right\}.
\]

Condition (1) ensures that the trajectory of the system remains in the neighborhood of the origin and so by integrating (6), we have:

\[
\int_{t+T}^{\infty} q(x(t'), k(x(t')) dt' \leq V(x_{t+T})
\]

that means the terminal cost functional \( V \) is an upper bound on the cost-to-go. Substituting (8) into (7) yields:

\[
J^*_\infty(x, t) \leq J^*(x, t, T).
\]

This relationship states that the optimal cost of the infinite horizon problem is bounded by the optimal cost of the corresponding finite horizon problem.

4. Calculating terminal cost and terminal region

Although in the proposed predictive control scheme, the suitable use of terminal cost and terminal set guarantees the closed-loop asymptotic stability, the calculation of these parameters is not a trivial task. The aim of this section is to derive a structured approach to obtain the terminal cost and the terminal region for finite horizon problems (2)–(3). To avoid unnecessary computational complexity, hereafter the following system is considered:

\[
\dot{x}(t) = f(x(t), x(t - \tau_1) + \int_{t-\theta}^{t} G(\theta)F(x(t), x(t - \tau_1), \ldots, x(t' - \tau_m)) \dot{x}(t' + \theta) d\theta
\]

\[
x(t + \theta) \dot{x}(t) + g(x(t), x(t - \tau_1)) \mathbf{u}(t)
\]
in which $r < \tau$. In the following, simple conditions are introduced to calculate the terminal region and the terminal cost. The main idea consists of two parallel stages. First, utilizing the Razumikhin function $V_l = x(t)^T P x(t)$ with symmetric positive definite matrix $P$, a locally asymptotically stabilizing control law $u(t) = K x(t)$ is designed such that it renders a region of the form (10) invariant for the system (9):

$$
\Omega_\alpha = \left\{ x : \max_{\theta \in [-r, 0]} x(t + \theta)^T P x(t + \theta) \leq \alpha \right\},
$$

(10)

where $\alpha > 0$. Then, a Lyapunov–Krasovskii functional of the form (11) which satisfies $\dot{E}(x) \leq -q(x(t), K x(t))$ for all $x_t \in \Omega_\alpha$ is considered as the terminal cost:

$$
E(x(t)) = x(t)^T P x(t) + \int_{-r}^{0} x(t + \zeta)^T S x(t + \zeta) d\zeta
+ \int_{-r}^{0} \int_{0}^{t} x(t + \zeta)^T D(\theta) L(\theta) x(t + \zeta) d\zeta d\theta
$$

(11)

where $S$ and $L$ are symmetric positive definite matrices.

**Lemma 1.** If there exists symmetric positive definite matrices $M, M_1 : i = 1, 2, \ldots, 6$, $N_i : j = 1, 2$ and a matrix $N$ of appropriate dimensions satisfying the following LMI for every $\theta \in [-r, 0]$:

$$
\left[ \begin{array}{cccc}
L_{\theta} & A_1 BN & D(\theta) BN & 0 \\
N^T B^T A_1 & -\frac{r}{\tau_1} N_1 & 0 & 0 \\
N^T B^T D(\theta) & 0 & 0 & 1 \\
M - M_1 & 0 & 0 & 0 \\
\end{array} \right] < 0
$$

(12)

where $M - M_1 \leq 0$, $i = 1, 2, \ldots, 6$.

Then, control law $u(t) = N M^{-1} x(t)$ renders $\Omega_\alpha$ invariant for $P = M^{-1}$ and some $\alpha > 0$, where

$$
\left(1/\tau_1\right) \{ (A_0 + A_1) M + B N + M (A + A_1)^T + N^T B^T - \tau_1 A_0 M A_1 - \tau_1 A_1 M A_1^T - \tau_1 A_1 M A_1^T - \tau_1 A_1 M A_1^T \}
- \left( \begin{array}{c}
-x(t)^T Q x(t) + x(t)^T (t - 1) R K x(t)
\end{array} \right)
$$

The proof is given in Appendix.

As usual, the stage cost is defined as $q(x, u) = x(t)^T Q x(t) + u(t)^T R u(t)$ with given symmetric positive definite matrices $Q$ and $R$. In Lemma 2, BMI conditions are determined to deduce the unknown matrices in $E(x)$ such that $E(x) \leq -x(t)^T Q x(t) + x(t)^T (t - 1) R K x(t)$ holds.

**Lemma 2.** If there exists a matrix $N$, symmetric positive definite matrices $M, L, S$, and scalar $\epsilon$ such that the following bilinear matrix inequalities are satisfied:

$$
\begin{bmatrix}
Z & I & N & M & M \\
I & -\frac{1}{\epsilon} & 0 & 0 & 0 \\
\frac{1}{r} & 0 & 0 & 0 & 0 \\
N^T & -V & 0 & 0 & 0 \\
M & 0 & 0 & -W & 0 \\
M & 0 & 0 & 0 & -\frac{1}{\epsilon}
\end{bmatrix} \leq 0
$$

(13)

then, for local control law $u(t) = N M^{-1} x(t)$, the inequality (6) is met for every $x_t \in \Omega$, where $P = M^{-1}$ and $S = M^{-1} S M$. Moreover, $W = Q^{-1}, V = R^{-1}$ and $Z = A_0 M + M A_0^T + B N + N^T B^T + S + I$. The proof is given in Appendix.

**Remark 1.** If $L$ in (11) is chosen to be $\lambda I$, where $\lambda$ is positive scalar parameter, then the BMI are easily solvable. In general, the integral may be discretized and implemented as follows:

$$
\int_{-r}^{0} G(\theta) L(\theta) d\theta = h \sum_{i=0}^{p-1} G(-\tau + ih)L^T_\theta(-\tau + ih),
$$

where $p$ is a positive integer and $h = r/p$. The mentioned trick is also applicable for the term $\int_{-r}^{0} \{ D(\beta) M_\beta D(\beta) \} d\beta$ of $L_\theta$ in (12).

**Theorem 2.** If there are symmetric positive definite matrices $M, L, S_1, M_1 : i = 1, 2, \ldots, 6$, $N_i : j = 1, 2$, matrix $N$, all of appropriate dimensions and scalar $\epsilon$ satisfying inequalities in (12) and (13), then $\Omega_\alpha$ in the form of (10) for some $\alpha > 0$, is selected as the terminal region and functional (11) is chosen as the terminal cost for the infinite horizon problem (2)-(3) with $q(x, u) = x(t)^T Q x(t) + u(t)^T R u(t)$ to achieve asymptotic stability of system (1).

**Proof.** From Lemmas 1 and 2 the proof of Theorem 2 follows immediately.

**Remark 2.** To determine $\alpha$ in (10), first the $\alpha_1$ is found as large as possible such that $K x \in U$ for all $x_t \in \Omega_\alpha$. Afterwards, by reducing $\alpha \in (0, \alpha_1]$ form $\alpha_1$ to zero, the largest possible $\alpha$ is obtained such that it satisfies the inequalities $E_{\Omega}(x) + q(x, K x) \leq 0$ and $V_{\alpha_1} < 0$ for all $x_t \in \Omega_\alpha$. This task can be implemented via the semi-infinite feasibility problem which is solved by existing efficient algorithms [18].

**Remark 3.** Note that the heavy computation to design the stabilizing predictive controller using the proposed procedure in Theorem 2 and Remark 2 is performed off-line before the implementation of the control strategy.

5. Illustrative example

As an example for demonstrating the proposed method, consider a nonlinear system with a delayed state described by:

$$
\dot{x}(t) = 6x(t) + x^3(t - 1) + \int_{-1}^{0} 1.5x(t + \theta) d\theta \\
+ [5 + x(t - 1)] u(t),
$$

with input constraint $|u| \leq 4$. The stage cost is defined as $q(x(t), u(t)) = x^2(t) + 0.2u^2(t)$. The linearization of system about the origin is as follows:

$$
\dot{x}(t) = 6x(t) + \int_{-1}^{0} 1.5x(t + \theta) d\theta + 5u(t).
$$

So, using Theorem 2, the design parameters are obtained as $K = -5.57, P = 2.77, L = 0.45, S = 7.61$ and $\epsilon = 2$ for $\delta = 5$. Regarding (11), the terminal cost is written as follows:

$$
E(x) = 2.77x^2(t) + \int_{-1}^{0} 7.61x^2(t + \zeta) d\zeta \\
+ \int_{-1}^{0} \int_{\theta}^{0} 1.01x^2(t + \zeta) d\zeta d\theta.
$$
For any vectors $v_1, v_2 \in \mathbb{R}^n$ and any symmetric positive definite matrix $M \in \mathbb{R}^{n \times n}$, the inequality $2v_1^T v_2 \leq v_1^T M v_1 + v_2^T M^{-1} v_2$ holds.

Proof of Lemma 1. The common way to design a local control law is to consider Jacobian linearization of the nonlinear system. The linearization of (9) about the origin $0 \in C_1$ is as follows:

$$
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + \int_{-\tau}^{0} D(\theta) x(t + \theta) d\theta + B u(t),
$$

where: $A_0 = \partial f/\partial x(0)$, $A_1 = \partial f/\partial x(t - \tau_1)(0)$, $D = G(\theta) \partial F/\partial x(t + \theta)(0)$, and $B = g(0)$. Now, let $\Phi$ be the difference between nonlinear system (9) and its Jacobian linearization (14):

$$
\Phi(x, u) = f(x(t), x(t - \tau_1)) + \int_{-\tau}^{0} G(\theta) F(x(t), x(t - \tau_1), x(t + \theta)) d\theta + B u(t)

- \left[ A_0 x(t) + A_1 x(t - \tau_1) + \int_{-\tau}^{0} D(\theta) x(t + \theta) d\theta + B u(t) \right].
$$

Since $f$ is continuously differentiable and $\Omega_0$ in (10) is compact, there exist scalars $c_0, c_1$ and $c_0$ such that $\forall x(t) \in \Omega_0$ and asymptotically stabilizing control $u(t) = K x(t)$, the relation (15) holds:

$$
|\Phi(x, K x(t))| \leq c_0(\alpha) |x(t)| + c_1(\alpha) |x(t - \tau_1)| + \int_{-\tau}^{0} c_0(\alpha) |x(t + \theta)| d\theta.
$$

Since $\partial f/\partial x(0)$ is compact, $\partial f/\partial x(t - \tau_1)(0)$ and $G(\theta) \partial F/\partial x(t + \theta)(0)$, and $\partial f/\partial x(t - \tau_1)(0)$ is compact, $G(\theta) \partial F/\partial x(t + \theta)(0)$ is compact, the terminal region is obtained.

6. Conclusion

In this paper, we have developed a receding horizon predictive control scheme that guarantees the closed loop stability of constrained nonlinear systems with discrete and distributed delays in states through the use of a appropriate terminal cost functional and terminal region. Lyapunov–Razumikhin and Lyapunov–Krasovskii theorems are combined to present a straightforward procedure for the determination of the terminal cost and the terminal region. Provided that the Jacobian linearization of the nonlinear time-delay system is stabilizable, by solution of bilinear matrix inequalities (BMIs) the mentioned parameters are obtained. The proposed scheme has been successfully applied to an example.

Acknowledgements

The authors gratefully acknowledge Marcus Reble, Ulrich Münz and Prof. Frank Allgöwer from Institute for System Theory and Automatic Control (IST) at Stuttgart University for the constructive and valuable comments and stimulating discussions.

Appendix

For the sake of brevity, only the sketch of proofs is introduced here. But first a useful lemma is noticed which will be used to derive the results.

Lemma 3. For any vectors $v_1, v_2 \in \mathbb{R}^n$ and any symmetric positive definite matrix $M \in \mathbb{R}^{n \times n}$, the inequality $2v_1^T v_2 \leq v_1^T M v_1 + v_2^T M^{-1} v_2$ holds.

Proof of Lemma 1. The common way to design a local control law is to consider Jacobian linearization of the nonlinear system. The linearization of (9) about the origin $0 \in C_1$ is as follows:

$$
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + \int_{-\tau}^{0} D(\theta) x(t + \theta) d\theta + B u(t),
$$

where: $A_0 = \partial f/\partial x(0)$, $A_1 = \partial f/\partial x(t - \tau_1)(0)$, $D = G(\theta) \partial F/\partial x(t + \theta)(0)$, and $B = g(0)$. Now, let $\Phi$ be the difference between nonlinear system (9) and its Jacobian linearization (14):

$$
\Phi(x, u) = f(x(t), x(t - \tau_1)) + \int_{-\tau}^{0} G(\theta) F(x(t), x(t - \tau_1), x(t + \theta)) d\theta + B u(t)

- \left[ A_0 x(t) + A_1 x(t - \tau_1) + \int_{-\tau}^{0} D(\theta) x(t + \theta) d\theta + B u(t) \right].
$$

Since $f$ is continuously differentiable and $\Omega_0$ in (10) is compact, there exist scalars $c_0, c_1$ and $c_0$ such that $\forall x(t) \in \Omega_0$ and asymptotically stabilizing control $u(t) = K x(t)$, the relation (15) holds:

$$
|\Phi(x, K x(t))| \leq c_0(\alpha) |x(t)| + c_1(\alpha) |x(t - \tau_1)| + \int_{-\tau}^{0} c_0(\alpha) |x(t + \theta)| d\theta.
$$

Furthermore, $c_0, c_1, c_0 \to 0$ as $\alpha \to 0$. Using the following relation

$$
x(t - h) = x(t) - \int_{-h}^{0} \dot{x}(t + \alpha) d\alpha
$$

the representation of (9) in region $\Omega$ is changed in order to obtain finally delay-dependent conditions. A Razumikhin function candidate is defined as $V_1(x) = x^T(t) P x(t)$ where $P$ is symmetric positive definite matrix. The time derivative of $V_1$ along trajectories of nonlinear system is taken. Using Lemma 3 and Razumikhin condition which states $\delta^2(t + \alpha) \leq \delta^2(t) \|P\|_2$, for some $\delta > 1$ and $a \in [-2 \tau_1, 0]$, $V_1$ can be written as follows:

$$
\dot{V}_1 = \int_{-\tau}^{0} \xi(t) A \xi(t) d\theta + (\Sigma_1 + \Sigma_2 + \Sigma_2) |\xi|^2,
$$

where $A = L_x - (\tau_1/r) A_0 B M^{-1} B^T A_0 + (D(\theta)) B M^{-1} B^T D(\theta) + L_y$, $L_x$, $L_y$, $M^{-1}$ were introduced in (12). Moreover:
\[ \Sigma_1 = 2 \| P \| \left( c_0 + \delta \left( c_1 + \int_{-r}^{0} c_0 d\theta \right) \right) \]

\[ \Sigma_2 = -\tau_1 \| P \| \delta^2 \left( c_0 + c_1 + \int_{-r}^{0} c_0 d\theta \right)^2 \]

\[ \Sigma_3 = -\frac{r^2}{2} \| P \| \delta^2 \left( c_0 + c_1 + \int_{-r}^{0} c_0 d\theta \right)^2 . \]

To ensure \( V_1 < 0 \) in \( \Omega_\alpha \), \( \Lambda \) is considered to be negative and \( \alpha \) is selected small enough to satisfy:

\[ \Sigma_1 + \Sigma_2 + \Sigma_3 < -\int_{-r}^{0} \lambda_{\text{min}}(A) d\theta . \]

The choice of such \( \alpha \) is possible because \( c_0, c_1, c_0 \to 0 \) as \( \alpha \to 0 \). Thus by the Razumikhin theorem, it follows that \( \Omega_\alpha \) is invariant for system (9) with \( u(t) = Kx(t) \). The final stage is to transform the condition \( \Lambda < 0 \) to the form (12) by some manipulation and utilizing the Schur complement [17]. □

**Proof of Lemma 2.** The derivative of \( E \) in (11) along the solution of (9) when using \( u(t) = Kx(t) \) in \( \Omega_\alpha \) is as follows:

\[
\dot{E}(x) = \dot{x}^T(t) \left( PA_0 + A_0^T P + PBK + K^T B^T P \right) x(t) + 2 \dot{x}^T(t) P \int_{-r}^{0} D(\theta) x(t + \theta) d\theta + 2 \dot{x}^T(t) P \Phi(x(t), Kx(t)) + \dot{x}^T(t) Sx(t) - \dot{x}^T(t - \tau_1) Sx(t - \tau_1) + \int_{-r}^{0} \left[ \dot{x}^T(t) D^T(\theta) LD(\theta) x(t) - \dot{x}^T(t + \theta) D^T(\theta) LD(\theta) x(t + \theta) \right] d\theta . \tag{17}
\]

Utilizing Lemma 3, the inequalities (18) are obtained:

\[
2 \dot{x}^T(t) P A_0 x(t - \tau_1) \leq \dot{x}^T(t) P^2 x(t) + \dot{x}^T(t - \tau_1) A_0^T A_0 x(t - \tau_1) \]

\[
2 \dot{x}^T(t) P \int_{-r}^{0} D(\theta) x(t + \theta) d\theta \leq r \dot{x}^T(t) P L^{-1} P x(t) + \int_{-r}^{0} \dot{x}^T(t + \theta) D^T(\theta) LD(\theta) x(t + \theta) d\theta . \tag{18}
\]

Hereafter, it is assumed that the following holds:

\[
\int_{-r}^{0} G^T(\theta) L G(\theta) d\theta \leq \varepsilon I . \tag{19}
\]

By substituting (18) and (19) in (17), the inequality (6) can be written as follows:

\[
\dot{E} + q + \dot{x}^T(t) \left( A_0^T P + PA_0 + K^T B^T P + PBK + P^2 \right) + S + rP L^{-1} P + \varepsilon I + Q + K^T RK \leq 0 . \tag{20}
\]

To satisfy (6), it suffices to have:

\[
A_0^T P + PA_0 + K^T B^T P + PBK + P^2 \]

\[
+ P^2 + S + rP L^{-1} P + \varepsilon I + Q + K^T RK \leq 0 . \tag{21}
\]

And to select \( \alpha \) small enough. By a reasoning similar to the proof of Lemma 1, it can be shown that a suitable \( \alpha \) can be determined such that \( E + q \leq 0 \) holds. Now, let \( M = P^{-1} ; N = K^{-1} ; U = P^{-1} S P^{-1} ; V = R^{-1} ; W = Q^{-1} \); by repeated using of the Schur complement, inequalities (21) can be transformed to the form (13). □

**References**


