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The Stability of Constrained Receding Horizon Control

James B. Rawlings and Kenneth R. Muske

***Abstract*—An infinite horizon controller is developed that allows incorporation of input and state constraints in a receding horizon feedback strategy. For both stable and unstable linear plants, feasibility of the constraints guarantees nominal closed-loop stability for all choices of the tuning parameters in the control law. The constraints' feasibility can be checked efficiently with a linear program. It is always possible to remove state constraints in the early portion of the infinite horizon to make them feasible. The controller's implementation requires only the solution of finite dimensional quadratic programs.**

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I. Introduction

Receding horizon control, also known as moving horizon control and model predictive control, has become an attractive feedback strategy, especially for nonlinear plants or linear plants subject to input and state inequality constraints. For linear plants, the technique has been especially successful in the process industries [1], [2]. Mayne and Michalska [3] and Eaton and Rawlings [4] have proposed formulations for controlling nonlinear systems. Several investigators have developed stability analyses for linear plants with finite horizons and without constraints [5]–[7]. Sznaier and Damborg have developed certain classes of stabilizing modified receding horizon controllers for linear plants with certain types of input and state constraints [8]. Finally, Zafiriou has performed some analysis of the contraction properties of the closed loop for linear plants with finite horizons and with constraints [9].

The essence of receding horizon control is to determine a control profile that optimizes some open-loop performance objective on a time interval extending from the current time to the current time plus a prediction horizon. This control profile is implemented until a plant measurement becomes available. Feedback is incorporated by using the measurement to update the optimization problem for the next time step. In a critique of receding horizon control, Bitmead *et al.* [10] point out the poor stability properties of finite horizon control laws. One method of achieving stability with a finite horizon is to add a terminal constraint that the state be zero at the end of the horizon [11], [12]. Appending this constraint does give closed-loop stability, but the constraint is somewhat artificial since, in the closed-loop, it is not satisfied and the state only asymptotically approaches zero. Also, as discussed in Section II-B, the formulation with terminal state constraint is restricted to controllable systems, and the current approach will only require stabilizable systems.

Bitmead *et al.*, on the other hand, recommend infinite horizon laws and develop feedback gains in terms of Riccati matrix formulas. These laws then have guaranteed stability properties and are further used to develop adaptive controllers. Unfortunately, the constraint handling capability of receding horizon control is lost in this formulation. To handle constraints, the repeated optimization is required to determine which constraints will be active. The resulting controller is then nonlinear, even for a linear plant, and a feedback gain formulation is not possible. In a different context, Mayne and Michalska propose a formulation for nonlinear plants that includes switching to a stabilizing linear controller after the state is brought near the origin [13]. The objective function then consists of a finite horizon portion for the nonlinear system and an infinite horizon portion for the approximately linear system.

In this note, we formulate an infinite horizon problem but keep a finite number of decision variables in the optimization problem so it can be solved on line as a quadratic program. The note is organized as follows. Sections II-A and II-B discuss stability for unconstrained stable and unstable plants. Sections II-C-i and II-C-ii then present stability results for stable and unstable plants subject to input and state constraints. It is convenient to divide the discussion which follows into stable and

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unstable plant cases because the proofs of the theorems are different for the two cases and the solutions of the open-loop optimization problems are also different. Throughout it is assumed that the state is measured and the plant is known, linear and time invariant.

II. Results

The plant is given by

$$x_{k+1} = Ax_k + Bu_k, \qquad k = 0, 1, 2, \cdots$$

in which $x_k \in \Re^n$, $u_k \in \Re^m$, and x_0 is assumed measured. A convenient orthonormal basis for discrete time systems is the usual

$$\phi_k = \Big[0, 0, \cdots, 0, \underbrace{I_{m \cdot N}}_{k\text{th location}}, 0, 0, \cdots\Big]^T$$

which is complete in the space of square summable inputs. The N-dimensional projection of the input onto the basis is given by

$$u^N = [I_N \;\; 0] \sum_{k=0}^{N-1} \phi_k u_k \tag{1}$$

in which u_k is the control move at sample time k, $u_k = 0 \forall k \geq N$, and u^N is the $m \cdot N$ vector of the nonzero inputs in the horizon.

The open-loop problem is chosen to be minimization of the standard quadratic objective function on an infinite horizon

$$\min_{u^N} \Phi = \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^T R u_k) \tag{2}$$

in which Q and R are positive definite, symmetric weighting matrices. In the receding horizon framework only the first move u_0 is injected into the plant. At the next sample time, Problem 2 is resolved with the new measured state as its initial condition.

A. Stable Plants

For stable plants we have the following result.

Theorem 1: For stable A and $N \geq 1$, the receding horizon controller with objective function, (2) is stabilizing.

We require a few results for this controller's associated Riccati equation to prove the theorem. Since A is convergent, define

$$K = \sum_{i=0}^{\infty} A^{T^i} Q A^i.$$

The solution to Problem 2 is then

$$u^N = -(E_N^{-1} B_N^T G_N A) x_0$$

in which

$$E_N = \begin{bmatrix} B^TKB + R & B^TA^TKB & \cdots & B^TA^{T^{N-1}}KB \\ B^TKAB & B^TKB + R & \cdots & B^TA^{T^{N-2}}KB \\ \vdots & \vdots & \ddots & \vdots \\ B^TKA^{N-1}B & B^TKA^{N-2}B & \cdots & B^TKB + R \end{bmatrix} \tag{3}$$

$$G_N = \begin{bmatrix} K \\ KA \\ \vdots \\ KA^{N-1} \end{bmatrix} \qquad B_N = \begin{bmatrix} B & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{bmatrix}$$

E_N is nonsingular due to R being positive definite. Since only the first move is implemented, the feedback gain for the closed-loop controller is

$$F_N = -\begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} E_N^{-1} B_N^T G_N A. \tag{4}$$

Showing that $\rho_N = A + BF_N$ is convergent for $N \geq 1$ then proves the theorem. The following two lemmas provide convenient expressions to evaluate the feedback gain and the open-loop objective function in terms of a Riccati matrix, Π_N.

Lemma 1: The feedback gain in (4) can be expressed in terms of a Riccati matrix

$$F_N = -(R + B^T \Pi_N B)^{-1} B^T \Pi_N A. \tag{5}$$

The Riccati matrix satisfies the recursion relation

$$\Pi_N = Q + A^T \Big[\Pi_{N-1} - \Pi_{N-1} B (B^T \Pi_{N-1} B + R)^{-1} \cdot B^T \Pi_{N-1} \Big] A, \qquad N > 1 \tag{6}$$

$$\Pi_1 = K. \tag{7}$$

The proof follows from the partitioned matrix inversion theorem and is omitted for brevity.

Lemma 2: The optimal value of the objective function in (2) is

$$\Phi^N = x_0^T \Pi_{N+1} x_0, \qquad N \geq 0. \tag{8}$$

The proof is by induction and is omitted for brevity.

Since the objective function in (2) is positive for every u^N and $x_0 \neq 0$, Lemma 2 implies

$$\Pi_N > 0, \qquad N \geq 1.$$

Also since the optimization problem for the N move controller has the same degrees of freedom as the $N - 1$ move controller plus one more, the objective function can be no worse, which implies

$$\Pi_{N+1} \leq \Pi_N, \qquad N \geq 1. \tag{9}$$

Finally one can construct a Lyapunov equation for ρ_N and show

$$\rho_N^T \Pi_{N\rho_N} - \Pi_N = -Q - F_N^T R F_N - (\Pi_N - \Pi_{N+1}).$$

The first term on the right-hand side is negative definite, the second term is negative semidefinite because $R > 0$, and the last term is negative semidefinite from (9). Therefore, the Lyapunov equation for ρ_N has a negative definite right-hand side and $\Pi_N > 0$, which is necessary and sufficient for ρ_N to be convergent [14] and Theorem 1 is proved.[1]

B. Unstable Plants

For unstable plants we have the following result.

Theorem 2: For stabilizable $\{A, B\}$ with r unstable modes and $N \geq r$, the receding horizon controller with objective function, (2) is stabilizing.

The stability proof for the unstable plant rests on analyzing the solution of (2) for u^N in the form of (1). Partition the Jordan form of the A matrix into stable and unstable parts

$$A = TJT^{-1} = [T_1 \quad T_2] \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix}$$

in which J_1's eigenvalues are unstable. The modes, $z = T^{-1}x$, then satisfy

$$z_{k+1} = Jz_k + \tilde{B}u_k.$$

To have a bounded objective function, u^N must bring the unstable modes, $z_k^1 = \tilde{T}_1 x_k$ to be zero at $k = N$. The proof is by

[1] Alternatively one can use a result from [15], once the monotonicity of Π_N has been established in (9), to prove the stability.

contradiction. If the unstable modes are not brought to zero at $k = N$, they evolve uncontrolled after $k = N$ from some nonzero value. Since J_1 is unstable and the horizon is infinite, this produces an unbounded objective function. So for the unstable plant case, we can write the optimization problem (2) as an equivalent problem

$$\min_{u^N} \Phi \tag{10}$$

$$\text{subject to:} \quad z_N^1 = 0. \tag{11}$$

The constraints are feasible since $N \geq r$ and $\{A, B\}$ is stabilizable. At this point one could proceed to develop a proof similar in technique to the proof of Theorem 1. This method cannot be used for systems with inequality constraints, however, so we develop a different proof that will be useful in the constrained problems in Section II-C.

Let $x_{k+j|k}, u_{k+j|k}$ denote the optimal state and input at time $k + j$ computed as the solution to (10) at time k. The value of the objective function at time k is then

$$\Phi_k = x_k^T Q x_k + x_{k+1|k}^T Q x_{k+1|k} + x_{k+2|k}^T Q x_{k+2|k} + \cdots + u_k^T R u_k + u_{k+1|k}^T R u_{k+1|k} + \cdots + u_{k+N-1|k}^T R u_{k+N-1|k} \tag{12}$$

At time $k + 1$, the state's initial condition is[2] $x_{k+1} = x_{k+1|k}$ because input u_k is injected into the plant at time k. The input sequence

$$\{u_{k+1|k}, u_{k+2|k}, \cdots, u_{k+N-1|k}, 0\}$$

is feasible at time $k + 1$ because it zeros the unstable modes at time $k + N - 1$ and keeps them zero at $k + N$. Therefore, since the optimization at $k + 1$ does not have to keep the last move in the sequence zero, the value of the objective function at $k + 1$ can be no worse than the right hand of (12) minus the x_k and u_k terms, or

$$\Phi_k \geq x_k^T Q x_k + u_k^T R u_k + \Phi_{k+1} \tag{13}$$

The sequence $\{\Phi_k\}$ is therefore nonincreasing. It is bounded below by zero and therefore converges. From (13) the terms $x_k^T Q x_k$ and $u_k^T R u_k$ then must converge to zero for large k. Since Q and R are positive definite, both x_k and u_k converge to zero for large k and the controller is stabilizing.

Kwon and Pearson [11] prove the closed-loop stability of the receding horizon controller with the terminal constraint $x_N = 0$ and $N \geq n + 1$. Their result is slightly more restrictive than this infinite horizon approach since it requires system controllability instead of stabilizability.

C. Constraints

We consider input and state (or output) constraints

$$Du_k \leq d, \qquad k = 0, 1, \cdots \tag{14}$$

$$Hx_k \leq h, \qquad k = 1, 2, \cdots \tag{15}$$

in which $d \in \Re^p$, and $h \in \Re^q$, and $d_i, h_i > 0$. Let U denote the convex feasible region for the inputs defined in (14). The input constraints represent physical limitations such as valve saturations. The state constraints are used for states or outputs that may not have setpoints, but should be kept within certain limits for plant operation reasons. The input and state constraints are specified on the infinite horizon. We show later with an example why this is important for stability.

i) Stable Plants

For stable plants, the input constraints are feasible independent of $\{A, B\}$, x_0, and N. The input constraints can obviously be converted to a finite set because $u_k = 0$, $k \geq N$.

$$Du_k \leq d, \qquad k = 0, 1, \cdots, N - 1 \tag{16}$$

The state constraints may be infeasible, but they can be converted into a feasible set by removing them for small k.

$$Hx_k \leq h, \qquad k = k_1, k_1 + 1, \cdots \tag{17}$$

We show that finite k_1 exists such that (17) is feasible. Let $h_{\min} = \min_i h_i$ and $\lambda_{\max} = \max_i |\lambda_i(A)|$. Consider a zero input sequence. Since the plant is stable, for large enough k the states become arbitrarily small and the constraints are satisfied. A bound for k_1 can be derived,[3]

$$k_1 = \max\left\{\ln\left(\frac{h_{\min}}{\|H\| K(T) \|x_0\|}\right) \Big/ \ln(\lambda_{\max}), 1\right\} \tag{18}$$

in which $K(T)$ is the condition number of T. It is important to note that once an admissable k_1 is chosen, in the nominal case the constraint horizon can slide forward at each future sample time. Therefore, for the nominal model and no disturbances, the constraints would be satisfied in the closed loop for $k \geq k_1$.

Finally we show that the constraints need only be satisfied on a finite horizon to guarantee satisfaction on the infinite horizon. There exists finite k_2 such that

$$Hx_k \leq h, \quad k = k_1, \cdots k_2 \Rightarrow Hx_k \leq h, \quad \forall k > k_2.$$

Let x_N be achieved by some input sequence, u_k, $k = 0, \cdots, N - 1$. Simple bounding arguments lead to

$$\|Hx_{N+k}\| \leq \|H\| K(T) \lambda_{\max}^k \|x_N\|.$$

Therefore the following k_2 is sufficiently large,

$$k_2 = N + \max\left\{\ln\left(\frac{h_{\min}}{\|H\| K(T) \|x_N\|}\right) \Big/ \ln(\lambda_{\max}), 0\right\}. \tag{19}$$

With these preliminaries we can state the following result.

Theorem 3: For stable A and $N \geq 1$, $x_k = 0$ is an asymptotically stable solution of the closed-loop receding horizon controller with objective function (2) and feasible constraints, (16), (17) for every $x_0 \in \Re^n$.

Proof: The constraints' feasibility at $k = 0$, implies feasibility at every k since feasibility at some k implies the input sequence $\{u_{k+1|k}, u_{k+2|k}, \cdots, u_{k+N-1|k}, 0\}$ is feasible at $k + 1$. Therefore, by induction, each quadratic program in the receding horizon formulation has a solution. Since the constraints are feasible, stability can be argued the same way as in the proof of Theorem 2. Therefore, even in the presence of constraints, (13) holds, $\{\Phi_k\}$ converges, and x_k, u_k converge to zero giving asymptotic stability.

ii) Unstable Plants

For unstable plants, the input constraints' feasibility cannot be checked independently of $\{A, B\}$, x_0, and N. We proceed to define admissable x_0. Let X_N denote the set of x_0 for which there exists $\{u_k\}_0^{N-1} \in U$ and $u_k = 0$, $k \geq N$ such that $\lim_{k \to \infty} x_k = 0$. The system is constrained stabilizable if and only if $x_0 \in X_\infty$ (see [8, def. 2]). The system can be stabilized by controllers under consideration here if and only if $x_0 \in X_N$.

[2] Nominal plant, no disturbances.

[3] We assume for simplicity that the eigenvalues of A are distinct in the following.

Since $X_N \subseteq X_\infty$, $x_0 \in X_N$ is a sufficient condition for the system to be constrained stabilizable. If this condition is met, the state constraints can be handled as in the stable plant case. Consider an input sequence that zeros the unstable modes. Then one can derive the bound

$$\|Hx_{N+k}\| \leq \|H\| \, \|T_2\| \lambda_{\max}^k \|z_N^2\|$$

in which $\lambda_{\max} < 1$ is the largest magnitude eigenvalue of J_2 and $z_N^2 = \tilde{T}_2 x_N$ are the stable modes at $k = N$. Choosing k_1 so that

$$k_1 \geq N + \max\left\{\ln\left(\frac{h_{\min}}{\|H\| \, \|T_2\| \, \|z_N^2\|}\right) \Big/ \ln(\lambda_{\max}), 1\right\} \quad (20)$$

makes the state constraints feasible. The boundedness of k_2 follows similarly. So finally one can pose for unstable plants the following problem.

$$\min_{u_N} \Phi \quad (21)$$

subject to

$$z_k^1 = 0, \qquad k = N \quad (22)$$

$$Du_k \leq d, \qquad k = 0, 1, \cdots, N-1 \quad (23)$$

$$Hx_k \leq h, \qquad k = k_1, k_1 + 1, \cdots. \quad (24)$$

The main result for this case is summarized with the following theorem.

Theorem 4: For stabilizable $\{A, B\}$ with r unstable modes and $N \geq r$, $x_k = 0$ is an asymptotically stable solution of the closed-loop receding horizon controller for the feasible quadratic program, (21)–(24) for every $x_0 \in X_N$.

One can check that the conditions of the theorem are met by first checking the stabilizability condition (22) and (23) with a linear program. If the expected x_0 are not in X_∞, the plant should probably be redesigned because it cannot be stabilized. If the expected x_0 are in X_∞ but not in X_N one can enlarge X_N by increasing N. If this is impractical because of computational limits, the approach as outlined cannot be applied. The state constraints can always be made feasible by choosing k_1, and the state constraints are satisfied in the closed loop by the nominal system for $k \geq k_1$.

D. Example

We show a numerical example to illustrate some of the results. Consider the following stable plant,

$$A = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = [-2/3 \quad 1]$$

in which the input/output transfer function has an unstable zero at $z = 3/2$. An output constraint is included to keep $|y_k| \leq 0.5$. The state constraints are then

$$\begin{bmatrix} -2/3 & 1 \\ 2/3 & -1 \end{bmatrix} x_k \leq \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

The input is unconstrained and we choose[4]

$$Q = \begin{bmatrix} 4/9 & -2/3 \\ -2/3 & 1 \end{bmatrix}, \qquad R = 1, \qquad N = 5.$$

If this problem is formulated on a finite horizon, forcing the output to meet the constraints at small k causes the controller to invert the unstable zero and the closed loop is unstable as shown in Figs. 1 and 2 for $x_0 = [3 \ \ 3]^T$.[5] No choices of N, Q, and R can remedy this situation.[6] Notice that while the closed loop is unstable, the constraints are met at every finite k since the input is unconstrained.

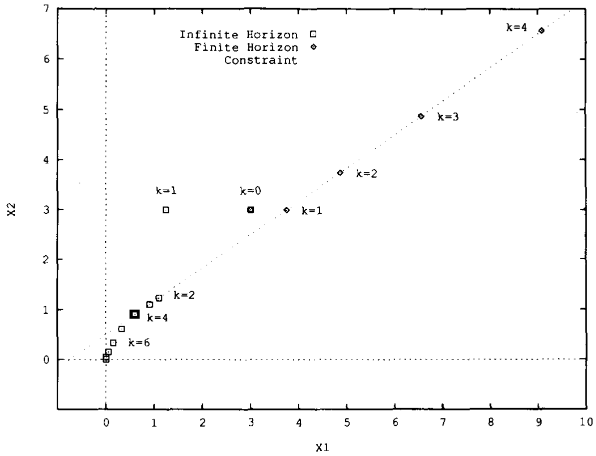

Fig. 1. State behavior for finite and infinite horizon controllers.

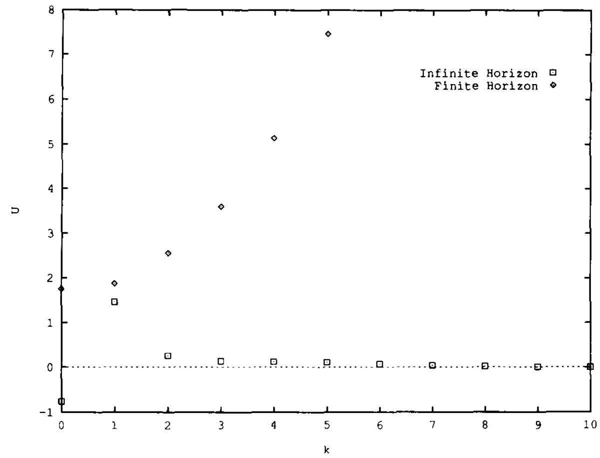

Fig. 2. Input versus time for finite and infinite horizon controllers.

Enforcing the constraints on the infinite horizon with too small k_1, however, makes the output constraints infeasible, which is what one would like the controller to diagnose. The problem is that the nonminimum phase behavior of the output imposes a limitation on the speed of the output response, and the constraints for small k attempt to violate this limitation. Increasing k_1 until $k_1 = n = 2$ makes the constraints feasible since there are no input constraints and the system is controllable. The infinite horizon controller with $k_1 = 2$ is stabilizing as shown in Fig. 1. Notice that the constraint is violated at $k = 1$ and is then satisfied for $k \geq 2$. In the calculation, the constraint

[4] Notice Q is positive semidefinite in this example. Extensions of Theorems 1–4 to handle this case are provided in [16].

[5] See [9] for other examples.

[6] To satisfy the constraint at the first sample time, $u_0 \geq 1.75$. Any input satisfying this inequality causes the states to increase at $k = 1$. This then forces $u_1 \geq u_0$ to meet the constraint at $k = 2$. The input and the states can be shown to be increasing sequences. The tuning parameters have no effect on this analysis.

was enforced on the infinite horizon by using $k_2 = 15$, which is sufficient according to (19).

III. Concluding Remarks

Since this formulation is close in structure to standard linear quadratic regulator theory, several extensions are possible. Time varying A, B, C, Q, R, d, and h could be considered. Tracking problems and nonzero setpoints can be treated in the standard way [17] with no special difficulties. The output feedback problem can be addressed by state estimation. The precise formulation of the filtering problem to preserve closed-loop stability is currently being studied. Finally, design for other performance goals that can be posed as optimization over tuning parameters N, Q, and R is an essentially unconstrained optimization, which should provide some advantages.

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On the Computation of the Gram Matrix in Time Domain and its Application

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Abstract—**The Gram matrix of the system is very useful in model-reduction applications. The computation of this matrix involves evaluation of scalar product of repeated integrals and can be computed in frequency domain as shown in [3], [10], [15]. In this note, a time-domain method for evaluation of the Gram matrix is presented. It is based on new properties of Gram matrix derived in this note. A new model reduction algorithm based on these properties is also presented. The method is finally illustrated by a numerical example.**

I. Introduction

The Gram matrix [5] has many applications namely system identification [5], [6], [8], [13], [12], modeling of power density spectrum [7], and model reduction [3], [10]. The Gram matrix contains elements which are scalar products of repeated integrals of impulse response of the system. In system identification application [5], the elements of this matrix can be generated experimentally, whereas in model-reduction application the elements of this matrix have to be computed from the mathematical model of the original system. This involves evaluation of integrals of the form

$$I_{j,k} = \int_0^{\infty} f_j(t) f_k(t)\, dt$$

where

$$f_1(t) = h(t) \quad \text{and} \quad f_{n+1}(t) = \int_{\infty}^{t} f_n(\tau)\, d\tau$$

$h(t)$ being the impulse response of the system. These integrals can be computed in frequency domain as suggested in [3], [10], [15]. These methods involve computing each and every element of the Gram matrix separately using different formulas. Although this poses no computational problems, it may be regarded by some as unattractive to use especially if the order of the system is very large.

In this note, a new time-domain method for computation of Gram matrix is proposed. It is based on the new properties of Gram matrix derived in the note. It is shown that the Gram matrix of a system is the impulse-response Gramian [1], [2], [16] of a reciprocal system and vice versa. It is also shown that the Gram matrix satisfies the Lyapunov equation with system matrices in a special form. A new model reduction algorithm based on these properties is also proposed. The proposed method involves reducing a reciprocal system instead of the original system using the method of [2] and reciprocating back the model obtained to get the required reduced-order model. This technique is shown match the first k time moments of the original system and the first $k \times k$ elements of the original system Gram matrix. The condition for asymptotic stability of reduced-order models is also derived. Further, the method proposed is very elegant and can

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