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Design and Stability Analysis of Fuzzy Model-based Predictive Control – A Case Study

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Abstract In the paper a fuzzy model based predictive control algorithm is presented. The proposed algorithm is developed in the state space and is given in analytical form, which is an advantage in comparison with optimisation based control schemes. Fuzzy model-based predictive control is potentially interesting in the case of batch reactors, heat-exchangers, furnaces and all the processes with strong nonlinear dynamics and high transport delays. In our case it is implemented to a continuous stirred-tank simulated reactor and compared to optimal PI control. Some stability and design issues of fuzzy model-based predictive control are also given.

Keywords fuzzy identification · predictive control · stability

1 Introduction

The fundamental methods which are essentially based on the principal of predictive control are Generalized Predictive Control [4], Model Algorithmic Control [21] and Predictive Functional Control [22], Dynamic Matrix Control [5], Extended Prediction Self-Adaptive Control [6] and Extended Horizon Adaptive Control [30]. All those methods are developed for linear process models. The principle is based on the process model output prediction and calculation of control signal which

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brings the output of the process to the reference trajectory in a way to minimise the difference between the reference and the output signal in a certain interval, between two prediction horizons, or to minimise the difference in a certain horizon, called coincidence horizon. The control signal can be found by means of optimisation or it can be calculated using the explicit control law formula [3, 11].

The nature of processes is inherently nonlinear and this implies the use of nonlinear approaches in predictive control schemes. Here, we can distinguish between two main group of approaches: the first group is based on the nonlinear mathematical models of the process in any form and convex optimisation [8], while the second group relies on approximation of nonlinear process dynamics with nonlinear approximators such as neural networks [28, 29], piecewise-linear models [20], Volterra and Wiener models [7], multi-models and multi-variables [16, 23], and fuzzy models [1, 25]. The advantage of the latter approaches is the possibility of stating the control law in the explicit analytical form.

In some highly nonlinear cases the use of nonlinear model-based predictive control can be easily justified. By introducing the nonlinear model into predictive control problem, the complexity increases significantly. In [3, 11], an overview of different nonlinear predictive control approaches is given.

When applying the model based predictive control with Takagi–Sugeno fuzzy model, it is always important how to choose fuzzy sets and corresponding membership functions. Many existing clustering techniques can be used in the identification phase to make this task easier. There exist many fuzzy model based predictive algorithms [2, 12, 24] that put significant stress on the algorithm that properly arranges membership functions.

In this work we are presenting a new fuzzy model based predictive control (FMBPC) algorithm in state-space form. The approach is an extension of predictive functional algorithm [22] to the nonlinear systems. The proposed algorithm easily copes with phase non-minimal and time-delayed dynamics. The control law in the case of FMBPC approach is given in analytical form. This makes the approach very easily implementable also programmable logic controllers and other hardware that is often used in the industrial practice. The approach can be combined by a clustering technique to obtain the FMBPC algorithm where membership functions are not fixed a priori but this is not intention of this work.

The paper is organised as follows. Section 2 describes the continuous stirred-tank reactor and identification of the process model in fuzzy form. In Section 3 the fuzzy model-based predictive control algorithm is introduced. Section 4 discusses stability and design issues and in Section 5 the simulation results are presented, discussed and compared to optimal PI control. Finally, the conclusions are given in Section 6.

2 Fuzzy Model of a Continuous Stirred-tank Reactor

The simulated continuous stirred-tank reactor (CSTR) process consists of an irreversible, exothermic reaction, $A \rightarrow B$, in a constant volume reactor cooled by a single coolant stream, which can be modelled by the following equation [19]:

$$\dot{C}_A^0 = \frac{q}{V}\left[C_{A0} - C_A^0\right] - k_0 C_A^0 \exp\left(\frac{-E}{RT}\right) \tag{1}$$

Table 1 Nominal CSTR parameter values

Quantity	Symbol	Value
Measured product concentration	C_A	0.1 mol/l
Reactor temperature	T	438.54 K
Coolant flow rate	q_c	$103.41\ \text{l min}^{-1}$
Process flow rate	q	$100\ \text{l min}^{-1}$
Feed concentration	C_{A0}	1 mol/l
Feed temperature	T_0	350 K
Inlet coolant temperature	T_{c0}	350 K
CSTR volume	V	100 l
Heat transfer term	hA	7×10^5 cal min^{-1} K^{-1}
Reaction rate constant	k_0	7.2×10^{10} min^{-1}
Activation energy term	E/R	1×10^4 K
Heat of reaction	ΔH	-2×10^5 cal/mol
Liquid densities	ρ, ρ_c	1×10^3 g/l
Specific heats	C_p, C_{pc}	1 cal g^{-1}K^{-1}

$$\dot{T} = \frac{q}{V}(T_0 - T) - \frac{k_0 \Delta H}{\rho C_p} C_A^0 \exp\left(\frac{-E}{RT}\right) + \frac{\rho_c C_{pc}}{\rho C_p V} q_c \times$$
$$\times \left[1 - \exp\left(-\frac{hA}{q_c \rho_c C_{pc}}\right)\right](T_{c0} - T) \tag{2}$$

The actual concentration C_A^0 is measured with a time delay $t_d = 0.5$ min:

$$C_A(t) = C_A^0(t - t_d) \tag{3}$$

The objective is to control the concentration of A (C_A) by manipulating the coolant flow rate q_c. This model is a modified version of the first tank of a two-tank CSTR example from [10]. In the original model, the time delay was zero.

The symbol q_c represents the coolant flow rate (manipulated variable) and the other symbols represent constant parameters whose values are defined in Table 1. The process dynamics are nonlinear due to the Arrhenius rate expression which describes the dependence of the reaction rate constant on the temperature (T). This is why the CSTR exhibits some operational and control problems. The reactor presents multiplicity behaviour with respect to the coolant flow rate q_c, i.e. if the coolant flow rate $q_c \in$ (11.1 l/min, 119.7 l/min) there are three equilibrium concentrations C_A. Stable equilibrium points are obtained in the following cases:

– $q_c > 11.1$ l/min $\Rightarrow$ stable equilibrium point 0.92 mol/l $< C_A <$ 1 mol/l,
– $q_c < 111.8$ l/min $\Rightarrow$ stable equilibrium point $C_A < 0.14$ mol/l (the point where $q_c \approx 111.8$ l/min is a Hopf Bifurcation point).

If $q_c \in$ (11.1 l/min, 119.7 l/min) there is also at least one unstable point for the measured product concentration C_A. From the above facts, one can see that the CSTR exhibits quite complex dynamics. In our application, we are interested in the operation in the stable operating point given by $q_c = 103.41\ \text{l min}^{-1}$ and $C_A =$ 0.1 mol/l.

2.1 Fuzzy Identification

Typical fuzzy model [26] is given in the form of rules

$$\mathbf{R}_j: \ \textbf{if } x_{p1} \textbf{ is } \mathbf{A}_{1,k_1} \textbf{ and } x_{p2} \textbf{ is } \mathbf{A}_{2,k_2} \textbf{ and } \ldots \textbf{ and } x_{pq} \textbf{ is } \mathbf{A}_{q,k_q} \textbf{ then } y = \phi_j(\mathbf{x}) \qquad j = 1, \ldots, m$$
$$k_1 = 1, \ldots, f_1 \quad k_2 = 1, \ldots, f_2 \quad \ldots \quad k_q = 1, \ldots, f_q \tag{4}$$

The q-element vector $\mathbf{x}_p^T = \left[x_{p1}, ..., x_{pq}\right]$ denotes the input or variables in premise, and variable y is the output of the model. With each variable in premise x_{pi} $(i = 1, \ldots, q)$, f_i fuzzy sets $(\mathbf{A}_{i,1}, \ldots, \mathbf{A}_{i,f_i})$ are connected, and each fuzzy set $\mathbf{A}_{i,k_i}$ $(k_i = 1, \ldots, f_i)$ is associated with a real-valued function $\mu_{A_{i,k_i}}(x_{pi}) : \mathbb{R} \to [0, 1]$, that produces membership grade of the variable x_{pi} with respect to the fuzzy set $\mathbf{A}_{i,k_i}$. To make the list of fuzzy rules complete, all possible variations of fuzzy sets are given in Eq. 4, yielding the number of fuzzy rules $m = f_1 \times f_2 \times \cdots \times f_q$. The variables x_{pi} are not the only inputs of the fuzzy system. Implicitly, the n-element vector $\mathbf{x}^T = [x_1, ..., x_n]$ also represents the input to the system. It is usually referred to as the consequence vector. The functions $\phi_j(\cdot)$ can be arbitrary smooth functions in general, although linear or affine functions are usually used.

The system in Eq. 4 can be described in closed form if the intersection of fuzzy sets is previously defined. The generalised form of the intersection is the so-called *triangular norm* (T-norm). In our case, the latter was chosen as algebraic product yielding the output of the fuzzy system

$$\hat{y} = \frac{\sum_{k_1=1}^{f_1} \sum_{k_2=1}^{f_2} \cdots \sum_{k_q=1}^{f_q} \mu_{A_{1,k_1}}(x_{p1})\, \mu_{A_{2,k_2}}(x_{p2}) \ldots \mu_{A_{q,k_q}}(x_{pq})\, \phi_j(\mathbf{x})}{\sum_{k_1=1}^{f_1} \sum_{k_2=1}^{f_2} \cdots \sum_{k_q=1}^{f_q} \mu_{A_{1,k_1}}(x_{p1})\, \mu_{A_{2,k_2}}(x_{p2}) \ldots \mu_{A_{q,k_q}}(x_{pq})} \tag{5}$$

It has to be noted that a slight abuse of notation is used in Eq. 5 since j is not explicitly defined as running index. From Eq. 4 is evident that each j corresponds to the specific variation of indexes k_i, $i = 1, \ldots, q$.

To simplify Eq. 5, a partition of unity is considered where functions $\beta_j(\mathbf{x}_p)$ defined by

$$\beta_j(\mathbf{x}_p) = \frac{\mu_{A_{1,k_1}}(x_{p1})\, \mu_{A_{2,k_2}}(x_{p2}) \ldots \mu_{A_{q,k_q}}(x_{pq})}{\sum_{k_1=1}^{f_1} \sum_{k_2=1}^{f_2} \cdots \sum_{k_q=1}^{f_q} \mu_{A_{1,k_1}}(x_{p1})\, \mu_{A_{2,k_2}}(x_{p2}) \ldots \mu_{A_{q,k_q}}(x_{pq})}, \quad j = 1, \ldots, m \tag{6}$$

give information about the fulfilment of the respective fuzzy rule in the normalised form. It is obvious that $\sum_{j=1}^{m} \beta_j(\mathbf{x}_p) = 1$ irrespective of $\mathbf{x}_p$ as long as the denominator of $\beta_j(\mathbf{x}_p)$ is not equal to zero (that can be easily prevented by stretching the membership functions over the whole potential area of $\mathbf{x}_p$). Combining Eqs. 5 and 6 and changing summation over k_i by summation over j we arrive to the following equation:

$$\hat{y} = \sum_{j=1}^{m} \beta_j(\mathbf{x}_p) \phi_j(\mathbf{x}) \tag{7}$$

The class of fuzzy models have the form of linear models, this refers to $\{\beta^j\}$ as a set of basis functions. The use of membership functions in input space with overlapping receptive fields provides interpolation and extrapolation.

Very often, the output value is defined as a linear combination of consequence states

$$\phi_j(\mathbf{x}) = \boldsymbol{\theta}_j^T \mathbf{x}, \quad j = 1, \ldots, m, \quad \boldsymbol{\theta}_j^T = \left[\theta_{j1}, \ldots, \theta_{jn}\right] \tag{8}$$

If Takagi–Sugeno model of the 0th order is chosen, $\phi_j(\mathbf{x}) = \theta_{j0}$, and in the case of the first order model, the consequent is $\phi_j(\mathbf{x}) = \theta_{j0} + \boldsymbol{\theta}_j^T \mathbf{x}$. Both cases can be treated by the model (8) by adding 1 to the vector $\mathbf{x}$ and augmenting vector $\boldsymbol{\theta}$ with θ_{j0}. To simplify the notation, only the model in Eq. 8 will be treated in the rest of the paper. If the matrix of the coefficients for the whole set of rules is written as $\boldsymbol{\Theta}^T = [\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_m]$ and the vector of membership values as $\boldsymbol{\beta}^T(\mathbf{x}_p) = \left[\beta^1(\mathbf{x}_p), \ldots, \beta^m(\mathbf{x}_p)\right]$, then Eq. 7 can be rewritten in the matrix form

$$\hat{y} = \boldsymbol{\beta}^T(\mathbf{x}_p)\boldsymbol{\Theta}\mathbf{x} \tag{9}$$

The fuzzy model in the form given in Eq. 9 is referred to as affine Takagi–Sugeno model and can be used to approximate any arbitrary function that maps the compact set $\mathbf{C} \subset \mathbb{R}^d$ (d is the dimension of the input space) to $\mathbb{R}$ with any desired degree of accuracy [13, 28, 31]. The generality can be proven by Stone–Weierstrass theorem [9] which indicates that any continuous function can be approximated by fuzzy basis function expansion [17].

2.2 Fuzzy Identification of the Continuous Stirred-tank Reactor

From the description of the plant, it can be seen that there are two variables available for measurement – measured product concentration C_A and reactor temperature T. For the purpose of control it is certainly beneficial to make use of both although it is not necessary to feed back reactor temperature if one wants to control product concentration. In our case the simple discrete compensator was added to the measured reactor temperature output:

$$\Delta q_{c_{ff}} = K_{ff}\left[T(k) - T(k-1)\right], \tag{10}$$

where K_{ff} was chosen to be 3, while the sampling time $T_s = 0.1$ min. The above compensator is a sort of the D-controller that does not affect the equilibrium points of the system (the static curve remains the same), but it does to some extent affect their stability. In our case the Hopf bifurcation point moved from $(q_c, C_A) =$ (111.8 l/min, 0.14 mol/l) to $(q_c, C_A) =$ (116.2 l/min, 0.179 mol/l). This means that the stability interval for the product concentration C_A expanded from (0, 0.14 mol/l) to (0, 0.179 mol/l). The proposed FMBPC will be tested on the compensated plant, so we need fuzzy model of the compensated plant.

The plant was identified in a form of discrete second order model with the premise defined as $\mathbf{x}_p^T = \left[C_A(k)\right]$ and the consequence vector as $\mathbf{x}^T = \left[C_A(k), C_A(k-1), q_c(k - T_{D_m}), 1\right]$. The functions $\phi_j(\cdot)$ can be arbitrary smooth functions in general, although linear or affine functions are usually used. Due to strong nonlinearity the structure with six rules and equidistantly shaped gaussian membership functions was chosen. The normalised membership functions are shown in Fig. 1.

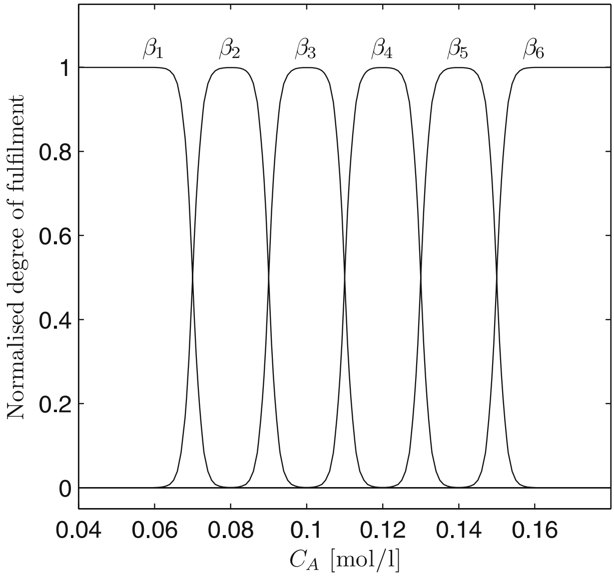

Fig. 1 The membership functions

The structure of the fuzzy model is the following:

$$\mathbf{R}_j: \textbf{ if } C_A(k) \textbf{ is } \mathbf{A}_j \textbf{ then } C_A(k+1) = -a_{1j}C_A(k) - a_{2j}C_A(k-1) + b_{1j}q_c(k-T_{D_m})+r_j \quad j=1,\ldots,6 \tag{11}$$

The parameters of the fuzzy form in Eq. 11 have been estimated using least square algorithm where the data have been preprocessed using QR factorisation [18]. The estimated parameters can be written as vectors $\mathbf{a}_1^T = [a_{11}, ..., a_{16}]$, $\mathbf{a}_2^T = [a_{21}, ..., a_{26}]$, $\mathbf{b}_1^T = \left[b_{11}, ..., b_{16}\right]$ and $\mathbf{r}_1^T = [r_{11}, ..., r_{16}]$. The estimated parameters in the case of CSTR are as follows:

$$\begin{aligned}
\mathbf{a}_1^T &= [-1.3462, -1.4506, -1.5681, -1.7114, -1.8111, -1.9157] \\
\mathbf{a}_2^T &= [0.4298, 0.5262, 0.6437, 0.7689, 0.8592, 0.9485] \\
\mathbf{b}_1^T &= [1.8124, 2.1795, 2.7762, 2.6703, 2.8716, 2.8500] \cdot 10^{-4} \\
\mathbf{r}_1^T &= [-1.1089, -1.5115, -2.1101, -2.1917, -2.5270, -2.7301] \cdot 10^{-2}
\end{aligned} \tag{12}$$

and $T_{D_m} = 5$.

After estimation of parameters, the TS fuzzy model (11) was transformed into the state space form to simplify the procedure of obtaining the control law:

$$\mathbf{x}_m(k+1) = \sum_i \beta_i(\mathbf{x}_p(k)) \left(\mathbf{A}_{m_i}\mathbf{x}_m(k) + \mathbf{B}_{m_i}u(k - T_{D_m}) + \mathbf{R}_{m_i}\right) \tag{13}$$

$$y_m(k) = \sum_i \beta_i(\mathbf{x}_p(k))\mathbf{C}_{m_i}\mathbf{x}_m(k) \tag{14}$$

$$\mathbf{A}_{m_i} = \begin{bmatrix} 0 & 1 \\ -a_{2_i} & -a_{1_i} \end{bmatrix} \quad \mathbf{B}_{m_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{C}_{m_i} = \begin{bmatrix} b_{1_i} & 0 \end{bmatrix} \quad \mathbf{R}_{m_i} = \begin{bmatrix} 0 \\ r_i/b_i \end{bmatrix} \tag{15}$$

where the process measured output concentration C_A is denoted by y_m and the input flow q_c by u.

The frozen-time theory [14, 15] enables the relation between the nonlinear dynamical system and the associated linear time-varying system. The theory establishes the following fuzzy model

$$\begin{aligned}\mathbf{x}_m(k+1) &= \bar{\mathbf{A}}_m\mathbf{x}_m(k) + \bar{\mathbf{B}}_m u(k - T_{D_m}) + \bar{\mathbf{R}}_m \\ y_m(k) &= \bar{\mathbf{C}}_m\mathbf{x}_m(k)\end{aligned} \tag{16}$$

where $\bar{\mathbf{A}}_m = \sum_i \beta_i(\mathbf{x}_p(k))\mathbf{A}_{m_i}$, $\bar{\mathbf{B}}_m = \sum_i \beta_i(\mathbf{x}_p(k))\mathbf{B}_{m_i}$, $\bar{\mathbf{C}}_m = \sum_i \beta_i(\mathbf{x}_p(k))\mathbf{C}_{m_i}$ and $\bar{\mathbf{R}}_m = \sum_i \beta_i(\mathbf{x}_p(k))\mathbf{R}_{m_i}$.

3 Fuzzy Model-based Predictive Control

In the case of FMBPC the prediction of the plant output is given by its fuzzy model in the state-space domain. This is why the approach in the proposed form is limited to the open-loop stable plants. By introducing some modifications the algorithm can be made applicable also for the unstable plants.

The problem of delays in the plant is circumvented by constructing an auxiliary variable that serves as the output of the plant if there were no delay present. The so-called "undelayed" model of the plant will be introduced for that purpose. It is obtained by "removing" delays from the original ("delayed") model and converting it to the state space description:

$$\begin{aligned}\mathbf{x}_m^0(k+1) &= \bar{\mathbf{A}}_m\mathbf{x}_m^0(k) + \bar{\mathbf{B}}_m\mathbf{u}(k) + \bar{\mathbf{R}}_m \\ y_m^0(k) &= \bar{\mathbf{C}}_m\mathbf{x}_m^0(k)\end{aligned} \tag{17}$$

where $y_m^0(k)$ models the "undelayed" output of the plant.

The behaviour of the closed-loop system is defined by the reference trajectory which is given in the form of the reference model. The control goal is to determine the future control action so that the predicted output value coincides with the reference trajectory. The time difference between the coincidence point and the current time is called a coincidence horizon. It is denoted by H. The prediction is calculated under the assumption of constant future manipulated variables ($u(k) = u(k+1) = \ldots = u(k+H-1)$), i.e. the mean level control assumption is used. The H-step ahead prediction of the "undelayed" plant output is then obtained from Eq. 17:

$$y_m^0(k+H) = \bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m^H\mathbf{x}_m^0(k) + \left(\bar{\mathbf{A}}_m^H - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\left(\bar{\mathbf{B}}_m u(k) + \bar{\mathbf{R}}_m\right)\right) \tag{18}$$

The reference model is given by the first order difference equation

$$y_r(k+1) = a_r y_r(k) + b_r w(k) \tag{19}$$

where w stands for the reference signal. The reference model parameters should be chosen so that the reference model gain is unity. This is accomplished by fulfilling the following equation

$$(1-a_r)^{-1}\, b_r = 1 \tag{20}$$

The main goal of the proposed algorithm is to find the control law which enables the reference trajectory tracking of the "undelayed" plant output $y_p^0(k)$. In each time instant the control signal is calculated so that the output is forced to reach the reference trajectory after H time samples ($y_p^0(k+H) = y_r(k+H)$). The idea of FMBPC is introduced through the equivalence of the objective increment vector Δ_p and the model output increment vector Δ_m:

$$\Delta_p = \Delta_m \tag{21}$$

The former is defined as the difference between the predicted reference signal $y_r(k+H)$ and the actual output of the "undelayed" plant $y_p^0(k)$

$$\Delta_p = y_r(k+H) - y_p^0(k) \tag{22}$$

The variable $y_p^0(k)$ cannot be measured directly. Rather, it will be estimated by using the available signals:

$$y_p^0(k) = y_p(k) - y_m(k) + y_m^0(k) \tag{23}$$

It can be seen that the delay in the plant is compensated by the difference between the outputs of the "undelayed" and the "delayed" model. When the perfect model of the plant is available, the first two terms on the right hand-side of Eq. 23 cancel and the result is actually the output of the "undelayed" plant model. If this is not the case, only the approximation is obtained. The model output increment vector Δ_m is defined by the following formula:

$$\Delta_m = y_m^0(k+H) - y_m^0(k) \tag{24}$$

The following is obtained from Eq. 21 by using Eqs. 22 and 24, and introducing Eq. 18:

$$u(k) = g_0^{-1}\Bigg(\left(y_r(k+H) - y_p^0(k) + y_m^0(k)\right) - \bar{\mathbf{C}}_m \bar{\mathbf{A}}_m^H \mathbf{x}_m^0(k) - \bar{\mathbf{C}}_m \left(\bar{\mathbf{A}}_m^H - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1} \bar{\mathbf{R}}_m \Bigg) \tag{25}$$

where g_0 stands for:

$$g_0 = \bar{\mathbf{C}}_m \left(\bar{\mathbf{A}}_m^H - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1} \bar{\mathbf{B}}_m \tag{26}$$

The control law of FMBPC in analytical form is finally obtained by introducing Eq. 23 into Eq. 25:

$$u(k) = g_0^{-1}\Bigg(\left(y_r(k+H) - y_p(k) + y_m(k)\right) - \bar{\mathbf{C}}_m \bar{\mathbf{A}}_m^H \mathbf{x}_m^0(k) - \bar{\mathbf{C}}_m \left(\bar{\mathbf{A}}_m^H - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1} \bar{\mathbf{R}}_m \Bigg) \tag{27}$$

In the following it will be shown that the realizability of the control law (27) relies heavily on the relation between the coincidence horizon H and the relative degree of the plant ρ. In the case of discrete-time systems, the relative degree is directly related to the pure time-delay of the system transfer function. If the system is described in the state-space form, any form can be used in general, but the analysis is much simplified in the case of certain canonical descriptions. If the system is described in controllable canonical form in each fuzzy domain, then also the matrices $\bar{\mathbf{A}}_m$, $\bar{\mathbf{B}}_m$, and $\bar{\mathbf{C}}_m$ of the fuzzy model (16) take the controllable canonical form:

$$\bar{\mathbf{A}}_m = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\bar{a}_n & -\bar{a}_{n-1} & -\bar{a}_{n-2} & \dots & -\bar{a}_1 \end{bmatrix}, \quad \bar{\mathbf{B}}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$\bar{\mathbf{C}}_m = \begin{bmatrix} \bar{b}_n & \dots & \bar{b}_\rho & 0 & \dots & 0 \end{bmatrix}, \qquad \bar{\mathbf{R}}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{r} \end{bmatrix} \tag{28}$$

where $\bar{a}_j = \sum_{i=1}^{m} \beta_i a_{j_i}$, $j = 1, ..., n$, $\bar{b}_j = \sum_{i=1}^{m} \beta_i b_{j_i}$, $j = \rho, ..., n$, and $\bar{r} = \sum_{i=1}^{m} \beta_i (r_i / b_i)$, $j = 1, ..., n$, and where the parameters a_{j_i}, b_{j_i} and r_i are state-space model parameters defined as in Eq. 15. Note that the state-space system with matrices from Eq. 28 has relative degree ρ what is reflected in the form of matrix $\bar{\mathbf{C}}_m$ – last $(\rho - 1)$ elements are equal to 0 while $\bar{b}_\rho \neq 0$.

Proposition 1 *If the coincidence horizon H is lower than the plant relative degree ρ ($H < \rho$), then the control law (27) is not applicable.*

Proof By taking into account the form of matrices in Eq. 28, it can easily be shown that

$$\left(\bar{\mathbf{A}}_m^H - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1} \bar{\mathbf{B}}_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \bullet \\ \vdots \\ \bullet \end{bmatrix} \tag{29}$$

i.e. the first $(n - H)$ elements of the vector are zeros, then there is the element 1, followed by $(H - 1)$ arbitrary elements. It then follows from Eqs. 26 and 29 that $g_0 = 0$ if $\rho > H$, and consequently the control law cannot be implemented. □

The closed-loop system analysis makes sense only in the case of non-singular control law. Consequently, the choice of H is confined to the interval $[\rho, \infty)$.

4 Stability Analysis

The stability analysis of the proposed predictive control can be performed using an approach of linear matrix inequalities (LMI) proposed in [29] and [27] or it can be done assuming the frozen-time theory [14, 15] which discusses the relation between the nonlinear dynamical system and the associated linear time-varying system.

In our stability study we have assumed that the frozen-time system given in Eq. 16 is a perfect model of the plant, i.e. $y_p(k) = y_m(k)$ for each k. Next, it is assumed that there is no external input to the closed-loop system ($w = 0$) – an assumption often made when analysing stability of the closed-loop system. Even if there is external signal, it is important that it is bounded. This is assured by selecting stable reference model, i.e. $|a_r| < 1$. The results of the stability analysis are also qualitatively the same if the system operates in the presence of bounded disturbances and noise.

Note that the last term in the parentheses of the control law (27) is equal to $g_0\bar{r}$. This is obtained using Eqs. 26 and 28. Taking this into account and considering the above assumptions the control law (27) simplifies to:

$$u(k) = g_0^{-1}\left(-\bar{\mathbf{C}}_m\bar{\mathbf{A}}_m^H\mathbf{x}_m^0(k)\right) - \bar{r} \tag{30}$$

Inserting the simplified control law (30) into the model of the "undelayed" plant (17) we obtain:

$$\mathbf{x}_m(k+1) = \left(\bar{\mathbf{A}}_m - \bar{\mathbf{B}}_m g_0^{-1}\bar{\mathbf{C}}_m\mathbf{A}_m^H\right)\mathbf{x}_m^0(k) \tag{31}$$

The closed-loop state transition matrix is defined as:

$$\mathbf{A}_c = \bar{\mathbf{A}}_m - \bar{\mathbf{B}}_m g_0^{-1}\bar{\mathbf{C}}_m\bar{\mathbf{A}}_m^H \tag{32}$$

If the system is in the controllable canonical form, the second term on the right-hand side of Eq. 32 has non-zero elements only in the last row of the matrix, and consequently $\mathbf{A}_c$ is also in the Frobenius form. The interesting form of the matrix is obtained in the case $H = \rho$. If $H = \rho$, it can easily be shown that $g_0 = \bar{b}_\rho$ and that $\mathbf{A}_c$ takes the following form:

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & \cdots & 0 & -\frac{\bar{b}_n}{\bar{b}_\rho} & \cdots & -\frac{\bar{b}_{\rho+1}}{\bar{b}_\rho} \end{bmatrix} \tag{33}$$

The corresponding characteristic equation of the system is:

$$z^\rho\left(z^{n-\rho} + \frac{\bar{b}_{\rho+1}}{\bar{b}_\rho}z^{n-\rho-1} + \frac{\bar{b}_{\rho+2}}{\bar{b}_\rho}z^{n-\rho-2} + \ldots + \frac{\bar{b}_{n-1}}{\bar{b}_\rho}z + \frac{\bar{b}_n}{\bar{b}_\rho}\right) = 0 \tag{34}$$

The solutions of this equation are closed-loop system poles: ρ poles lie in the origin of the z-plane while the other $(n-\rho)$ poles lie in the roots of the polynomial

$\bar{b}_\rho z^{n-\rho} + \bar{b}_{\rho+1} z^{n-\rho-1} + \ldots + \bar{b}_{n-1} z + \bar{b}_n$. These results can be summarised in the following proposition:

Proposition 2 *When the coincidence horizon is equal to the relative degree of the model ($H = \rho$), then ($n - \rho$) closed-loop poles tend to open-loop plant zeros, while the rest (ρ) of the poles go to the origin of z-plane.*

The proposition states that the closed-loop system is stable for $H = \rho$ if the plant is minimum phase. When this is not the case, the closed-loop system would become unstable if H is chosen equal to ρ. In such case the coincidence horizon should be larger.

The next proposition deals with the choice of a very large coincidence horizon.

Proposition 3 *When the coincidence horizon tends to infinity ($H \to \infty$) and the open-loop plant is stable, the closed-loop system poles tend to open-loop plant poles.*

Proof The proposition can be proven easily. In the case of stable plants, $\mathbf{A}_m$ is a Hurwitz matrix that always satisfies:

$$\lim_{H \to \infty} \mathbf{A}_m^H = 0 \tag{35}$$

Combining Eqs. 32 and 35 we arrive at the final result

$$\lim_{H \to \infty} \mathbf{A}_c = \mathbf{A}_m \tag{36}$$

□

The three propositions give some design guidelines on choosing the coincidence horizon H. If $H < \rho$, the control law is singular and thus not applicable. If $H = \rho$, the closed-loop poles go to open-loop zeros, i.e. high-gain controller is being used. If H is very large, the closed-loop poles go to open-loop poles, i.e. low-gain controller is being used and the system is almost open-loop. If the plant is stable, the closed-loop system can be made stable by choosing coincidence horizon large enough.

5 Simulation Results

Reference tracking ability and disturbance rejection capability of the FMBPC control algorithm were tested experimentally on a simulated CSTR plant. The FMBPC was compared to the conventional PI controller.

In the first experiment the control system was tested for tracking the reference signal that changed the operating point from nominal ($C_A = 0.1$ mol/l) to larger concentration values and back, and then to smaller concentration values and back. The proposed FMBPC used the following design parameters: $H = 9$ and $a_r = 0.96$. The parameters of the PI controller were obtained by minimising the following criterium:

$$C_{PI} = \sum_{k=0}^{400} (y_r(k) - y_{PI}(k))^2 \tag{37}$$

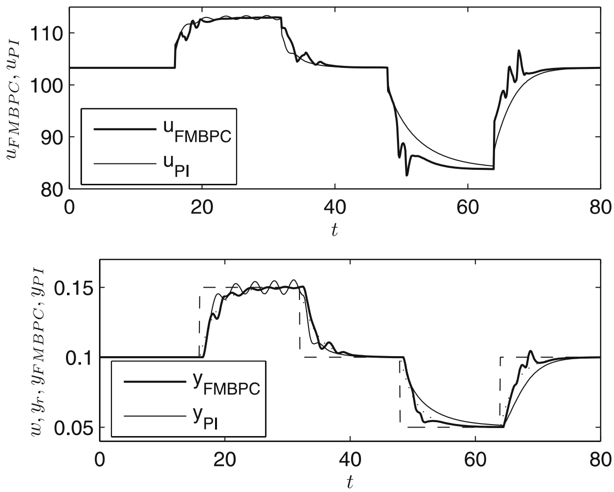

Fig. 2 The performance of the FMBPC and the PI control in the case of reference trajectory tracking

where $y_r(k)$ is the reference model output depicted in Fig. 2, and $y_{PI}(k)$ is the controlled output in the case of PI control. This means that the parameters of the PI controller were minimised to obtain the best tracking of the reference model output for the case treated in the first experiment. The optimal parameters were $K_P = 64.6454\ \mathrm{l^2 mol^{-1} min^{-1}}$ and $T_i = 0.6721$ min. Figure 2 also shows manipulated and controlled variables for the two approaches. In the lower part of the figure, the set-point is depicted with the dashed line, the reference model output with the dotted line, the FMBPC response with the thick solid line and the PI response with the thin solid line. The upper part of the figure represents the two control signals. The performance criteria obtained in the experiment are the following:

$$C_{PI} = \sum_{k=0}^{400} (y_r(k) - y_{PI}(k))^2 = 0.0165 \tag{38}$$

$$C_{FMBPC} = \sum_{k=0}^{400} (y_r(k) - y_{FMBPC}(k))^2 = 0.0061 \tag{39}$$

The disturbance rejection performance was tested with the same controllers that were set to the same design parameters as in the first experiment. In the simulation experiment, the step-like positive input disturbance of 3 l/min appeared and disappeared later. After some time, the step-like negative input disturbance of −3 l/min appeared and disappeared later. The results of the experiment are shown in Fig. 3 where the signals are depicted by the same line types as in Fig. 2. Similar performance criteria can be calculated as in the case of reference tracking:

$$C_{PI} = \sum_{k=0}^{400} (y_r(k) - y_{PI}(k))^2 = 0.0076 \tag{40}$$

$$C_{FMBPC} = \sum_{k=0}^{400} (y_r(k) - y_{FMBPC}(k))^2 = 0.0036 \tag{41}$$

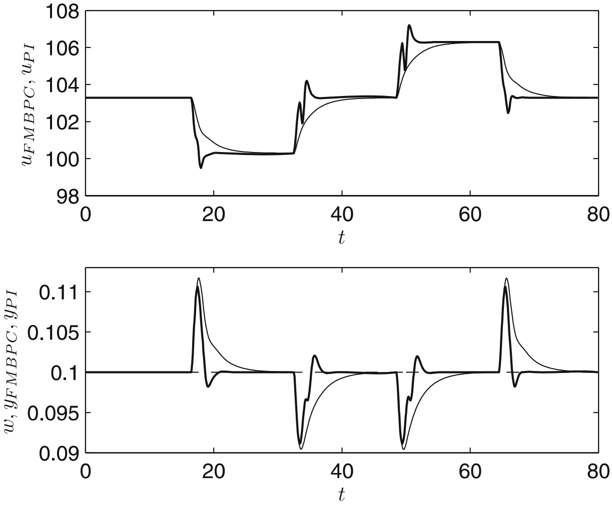

Fig. 3 The control performance of the FMBPC and the PI control in the case of disturbance rejection

The obtained simulation results have shown that better performance criteria are obtained in the case of the FMBPC control in both control modes: the trajectory tracking mode and the disturbance rejection mode. This is obvious because the PI controller assumes linear process dynamics, while the FMBPC controller takes into account the plant nonlinearity through the fuzzy model of the plant. The proposed approach is very easy to implement and gives a high control performance.

6 Conclusion

In the paper a new fuzzy model-predictive algorithm has been presented. The FMBPC control law is given in explicit analytical form and for that reason it is easily implementable in programmable logical controllers and other industrial hardware. The paper discusses stability issues of the proposed approach. The results of the stability study are some guidelines on choosing the design parameters of the controller. Fuzzy model based predictive control is potentially interesting in the case of batch reactors, heat-exchangers, furnaces and all the processes with strong nonlinear dynamics and high transport delays. In the paper the performance of the FMBPC control was tested on a simulated continuous stirred tank reactor and compared to the optimally tuned classical PI controller.

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